

Introduction to 2D Hydraulics Equations

Alex Sánchez, Ph.D.

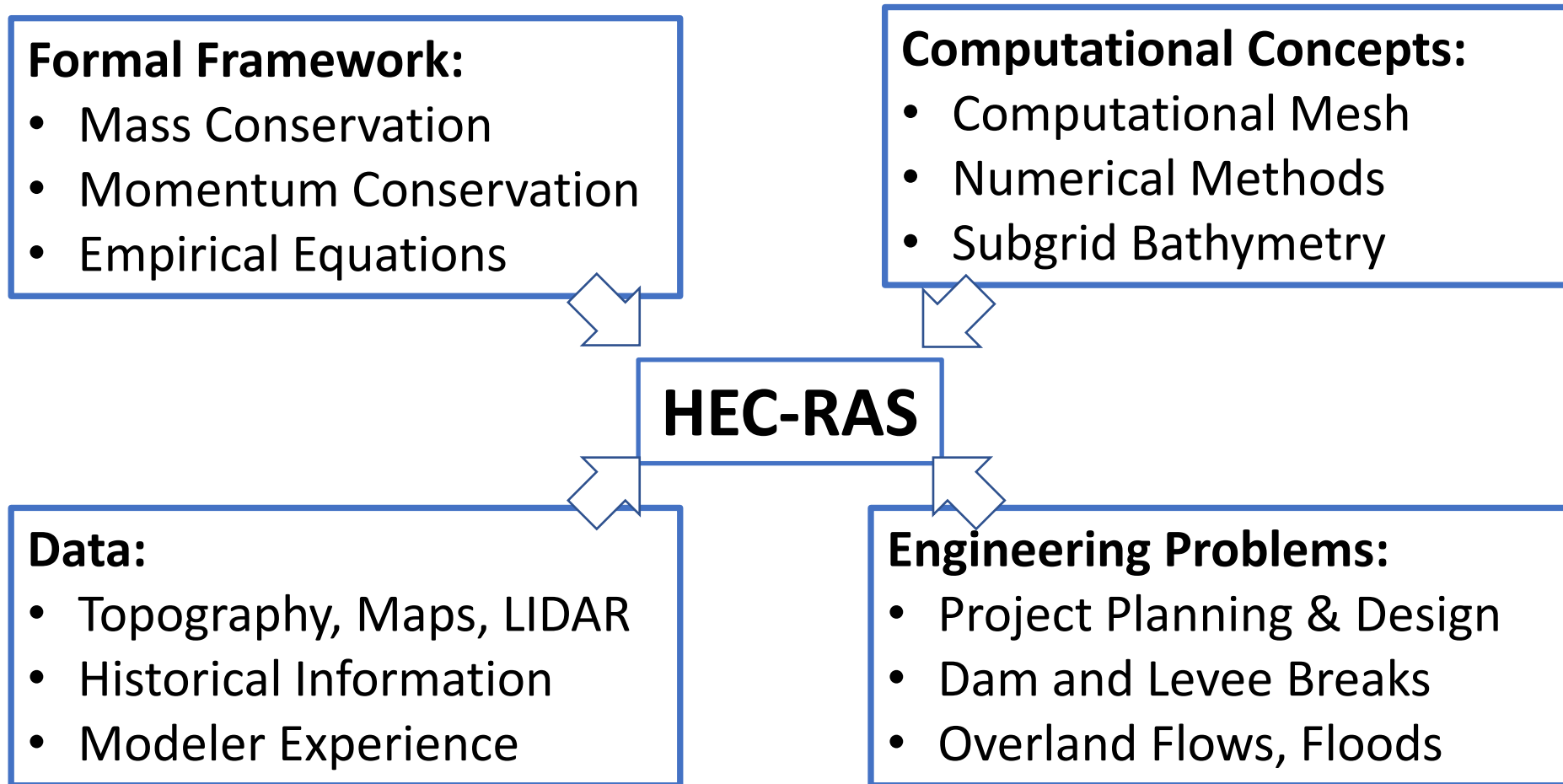
Senior Hydraulic Engineer

USACE, Institute for Water Resources, Hydrologic Engineering Center





Hydraulic Modeling





Outline

- Mass Conservation (Continuity)
- Momentum Conservation (Depth-Averaged)
 - Acceleration
 - Coriolis term
 - Hydrostatic pressure
 - Turbulent mixing
 - Friction
- Diffusion Wave Equation
- Numerical Methods

Mass Conservation

- Assuming a constant water density

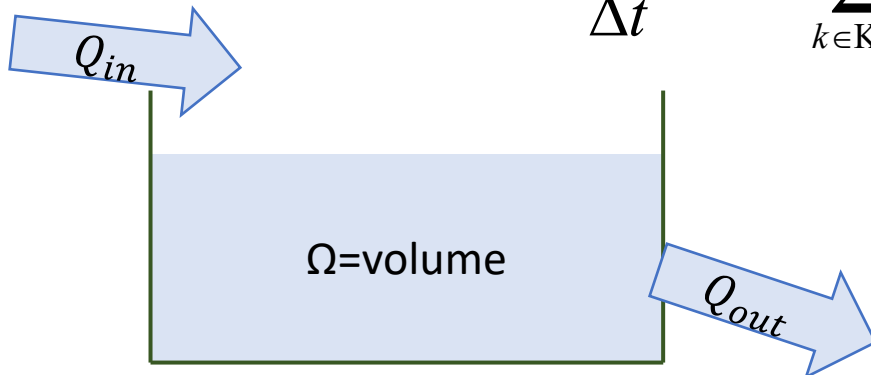
$$\frac{\partial h}{\partial t} + \nabla \cdot (h\mathbf{V}) = q$$

- Integrating over a computational cell

$$\frac{\partial}{\partial t} \iiint_{\Omega} d\Omega + \iint_S (\mathbf{V} \cdot \mathbf{n}) dS = Q$$

- Finite-Volume Discretization

$$\frac{\Omega_i^{n+1} - \Omega_i^n}{\Delta t} + \sum_{k \in K(i)} (\mathbf{V}_k \cdot \mathbf{n}_{ik}) A_k = Q_i$$



h : Water depth

q : Water source/sink

Ω_i : Cell water volume

A_k : Face area

\mathbf{V}_k : Face velocity

\mathbf{n}_{ik} : Outward face-normal unit vector

Δt : Time step

Change in volume in a system balances with flow through boundaries



Momentum Conservation

- Momentum Equation (non-conservative form)

$$\underbrace{\frac{\partial V}{\partial t}}_{\text{Temporal}} + \underbrace{(V \cdot \nabla)V}_{\text{Advection}} + \underbrace{f_c \mathbf{k} \times V}_{\text{Coriolis}} = - \underbrace{g \nabla z_s}_{\text{Pressure gradient}} + \underbrace{\frac{1}{h} \nabla \cdot (v_t h \nabla V)}_{\text{Diffusion}} - \underbrace{\frac{\tau_b}{\rho R}}_{\text{Bottom Friction}} + \underbrace{\frac{\tau_s}{\rho h}}_{\text{Wind Stress}}$$

- From Newton's 2nd Law of motion
- Assumes constant water density, small vertical velocities, hydrostatic pressure, etc.
- Non-linear and a function of both velocity and water levels
- Continuity and Momentum Equations are the Shallow Water Equations or sometimes referred to as the "Full Momentum" equations in HEC-RAS

V : Velocity

z_s : Water level

g : Gravity

v_t : Turbulent eddy
viscosity

h : Water depth

R : Hydraulic Radius

f_c : Coriolis Parameter

τ_b : Bed shear stress

τ_s : Surface stress



Acceleration and Total Derivative

- **Eulerian:** Frame of reference fixed in space and time

https://commons.wikimedia.org/wiki/File:Lagrangian_vs_Eulerian_frame_of_reference.svg

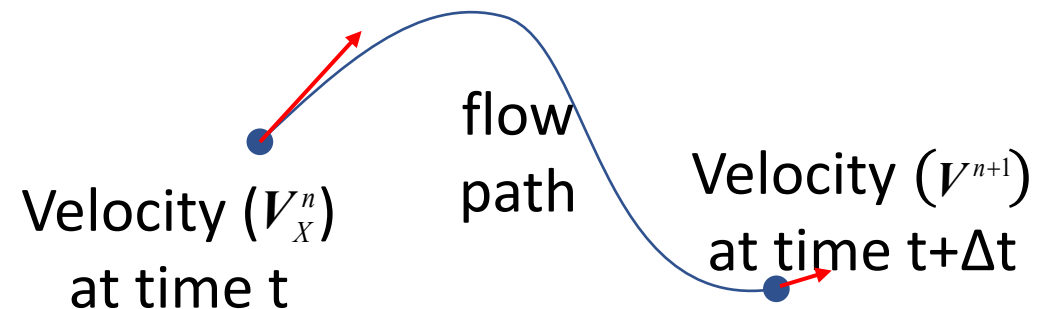
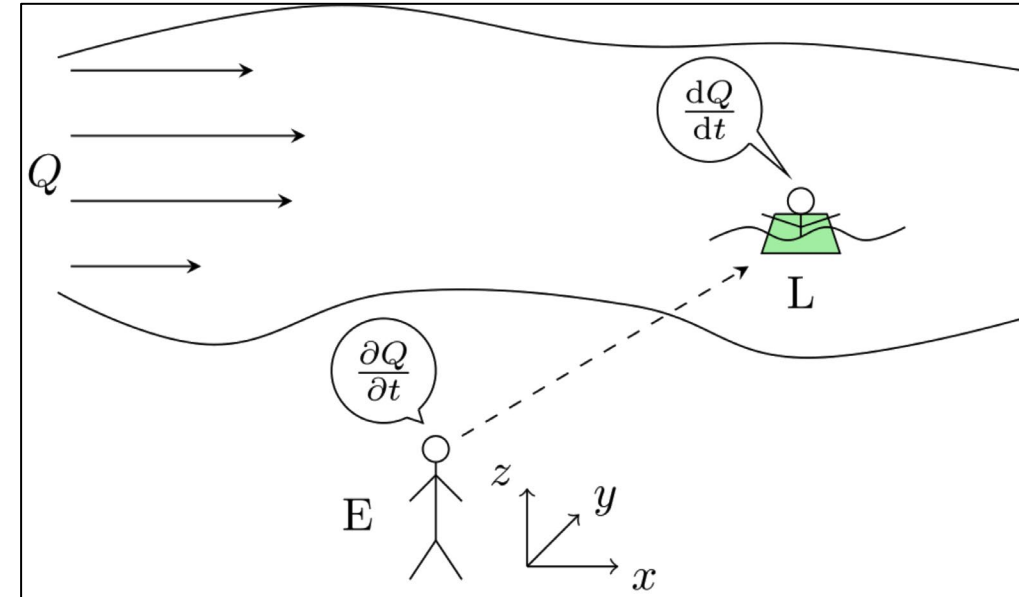
$$\frac{\partial V}{\partial t} + (V \cdot \nabla)V$$

- Easier to compute
- Time-step restricted by Courant condition

- **Lagrangian:** Frame of reference moves with total derivative along flow path

$$\frac{\partial V}{\partial t} + (V \cdot \nabla)V = \frac{DV}{Dt} = \frac{V^{n+1} - V_X^n}{\Delta t}$$

- More expensive to compute
- Allows larger time-steps



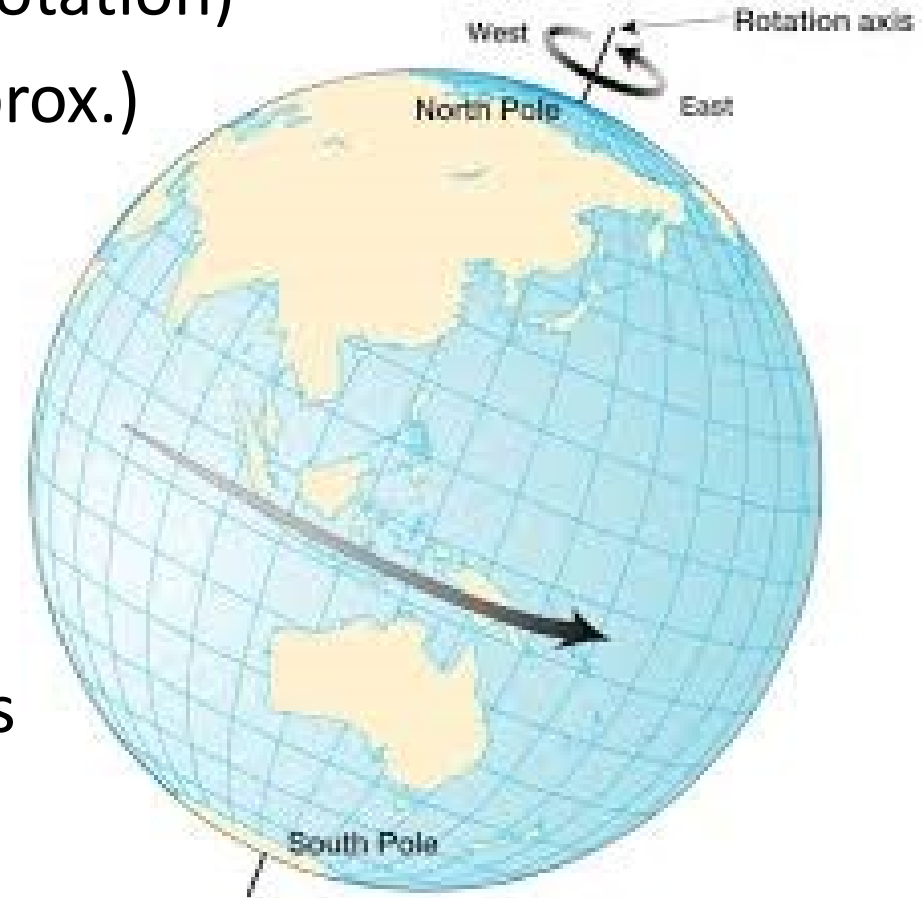


Coriolis Acceleration

- Effect of rotating frame of reference (earth's rotation)
- Constant for the each 2D domain (f-plane approx.)

$$f_c = 2 \omega \sin \varphi$$

- ω : sidereal angular velocity of the Earth
- φ : latitude. Positive for northern hemisphere. Negative for southern hemisphere
- Coriolis acceleration disabled by default to save computational time
- Negligible for most river and flood simulations
- When to enable Coriolis term?
 - Large domains
 - Higher latitudes





Hydrostatic Pressure

- Assumes vertical water accelerations are small compared to gravity
- Total pressure is

$$P = P_{atm} + \rho g(z_s - z)$$

- P_{atm} : atmospheric pressure (assumed to be constant)
 - ρ : constant water density
 - g : gravity acceleration constant
 - z_s : water surface elevation
 - z : vertical coordinate
- Pressure gradient

$$\frac{\partial P}{\partial x} = \rho g \frac{\partial z_s}{\partial x}$$



Diffusion of Momentum

- Non-conservative Formulation

- Only option in Version 5.0.7 and earlier,
- Optional in Version 6.0

$$\frac{DV}{Dt} = -g\nabla z_s + \boxed{v_t \Delta V} - \frac{\tau_b}{\rho R}$$

- Conservative Formulation

- Default in Version 6.0
- Only option for Eulerian SWE solver

$$\frac{DV}{Dt} = -g\nabla z_s + \boxed{\frac{1}{h} \nabla \cdot (v_t h \nabla V)} - \frac{\tau_b}{\rho R}$$

$\Delta = \nabla^2$: Laplacian

u_N : Face-normal velocity

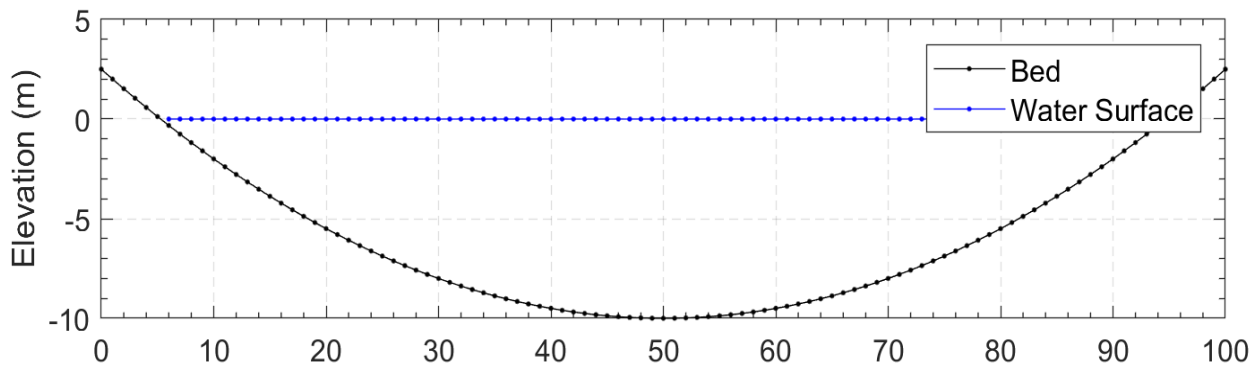
v_t : Turbulent eddy viscosity

h : Water depth

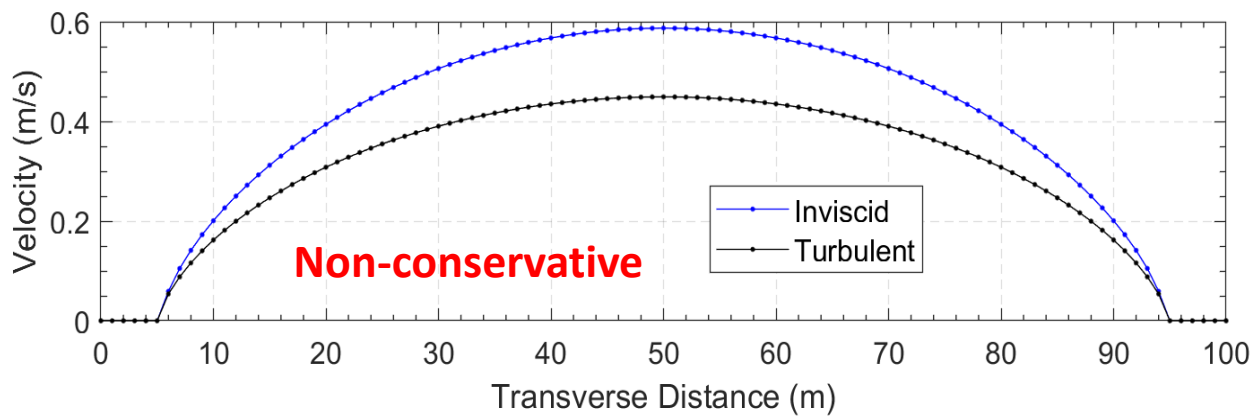
c_f : Non-linear friction coefficient



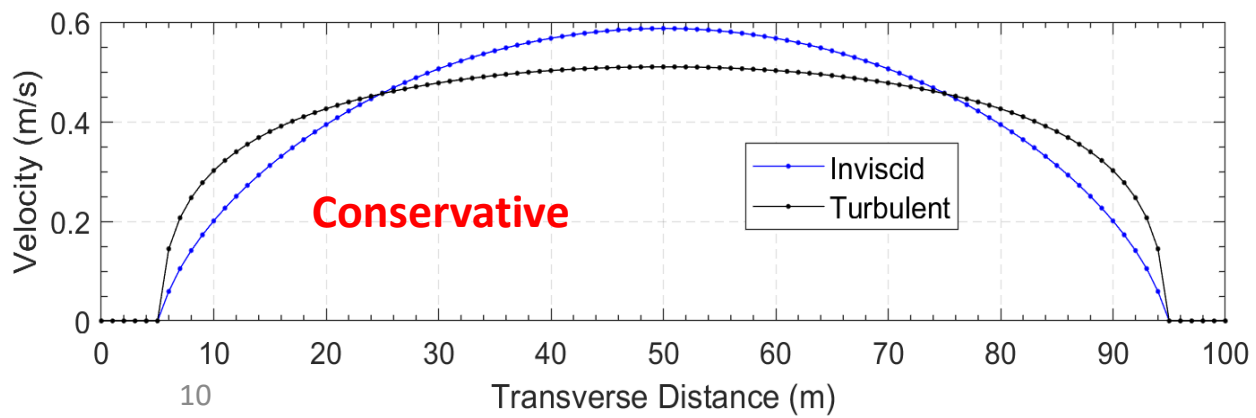
Mixing Term Formulation Comparison



Bathymetry and water level



Produces a net dissipation



Decreases velocities in middle of channel but increases velocities near banks



Eddy Viscosity: Turbulence Model

- Old: Parabolic $\nu_t = Du_*h$
 - Versions 5.0.7 and earlier
 - Isotropic (same in all directions)
 - 1 parameter: mixing coefficient D
- New: Parabolic-Smagorinsky

u_* : Shear velocity

h : Water depth

D : Mixing coefficient

D_L : Longitudinal mixing coefficient

D_T : Transverse mixing coefficient

C_s : Smagorinsky coefficient

$$\nu_t = \mathbf{D}u_*h + (C_s\Delta)^2 |\bar{S}|$$

$$|\bar{S}| = \sqrt{2\left(\frac{\partial u}{\partial x}\right)^2 + 2\left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2} \quad \mathbf{D} = \begin{bmatrix} D_{xx} & 0 \\ 0 & D_{yy} \end{bmatrix} \quad \begin{aligned} D_{xx} &= D_L \cos^2 \theta + D_T \sin^2 \theta \\ D_{yy} &= D_L \sin^2 \theta + D_T \cos^2 \theta \end{aligned}$$

- Default method in Version 6.0
- Non-Isotropic (not the same in all directions)
- 3 parameters: D_L , D_T , and C_s



Bottom Friction

- Resisting force due to relative motion of fluid against the bed
- Bed Shear Stress

$$\tau_b = \rho C_D |\mathbf{V}| \mathbf{V}$$

- Drag Coefficient

$$C_D = \frac{gn^2}{R^{1/3}}$$

- Friction coefficient

$$c_f = \frac{C_D}{R} |\mathbf{V}| = \frac{gn^2}{R^{4/3}} |\mathbf{V}|$$

n :Manning coefficient

ρ : water density

g : gravity acceleration constant

$|\mathbf{V}|$: velocity magnitude

R : hydraulic radius

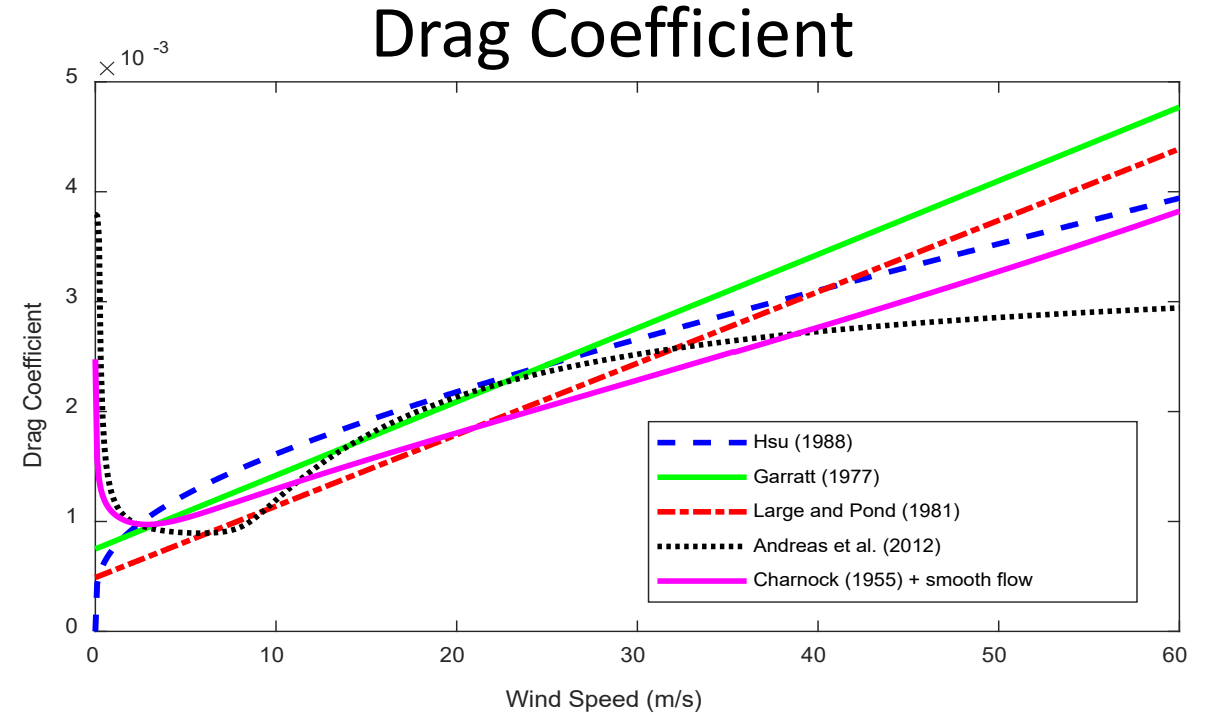
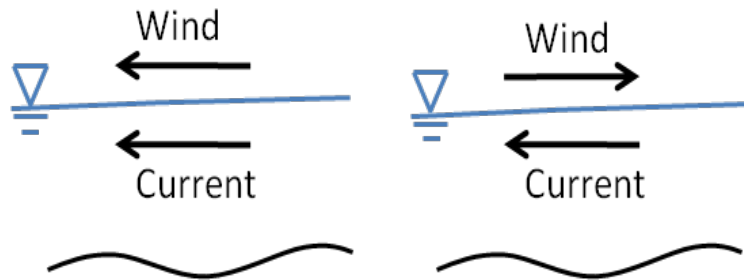
Wind Stress

- Surface Stress is given by

$$\tau_s = \rho_a C_D |W_{10}| W_{10}$$

- Wind Reference Frame

$$W_{10} = \begin{cases} W_{10}^E - V & \text{for Lagrangian} \\ W_{10}^E & \text{for Eulerian} \end{cases}$$





Diffusive-Wave Approximation

- Ignoring the following terms

$$\underbrace{\frac{\partial \mathbf{V}}{\partial t}}_{\text{Temporal}} + \underbrace{(\mathbf{V} \cdot \nabla) \mathbf{V}}_{\text{Advection}} + \underbrace{f_c \mathbf{k} \times \mathbf{V}}_{\text{Coriolis}} = \underbrace{-g \nabla z_s}_{\text{Pressure gradient}} + \underbrace{\frac{1}{h} \nabla \cdot (\mathbf{v}_t h \nabla \mathbf{V})}_{\text{Diffusion}} - \underbrace{\frac{\tau_b}{\rho R}}_{\text{Bottom Friction}} + \underbrace{\frac{\tau_s}{\rho h}}_{\text{Wind Stress}}$$

- Expanding and dividing both sides by the square of its norm leads to

$$\mathbf{V} = -\frac{\beta}{h} \nabla z_s \quad \beta = \frac{R^{2/3} h}{n |\nabla z_s|^{1/2}}$$

- Inserting the above equation into the Continuity Equation leads to the Diffusion-Wave Equation (DWE)

$$\frac{\partial h}{\partial t} = \nabla \cdot (\beta \nabla z_s) + q$$



SWE vs. DWE

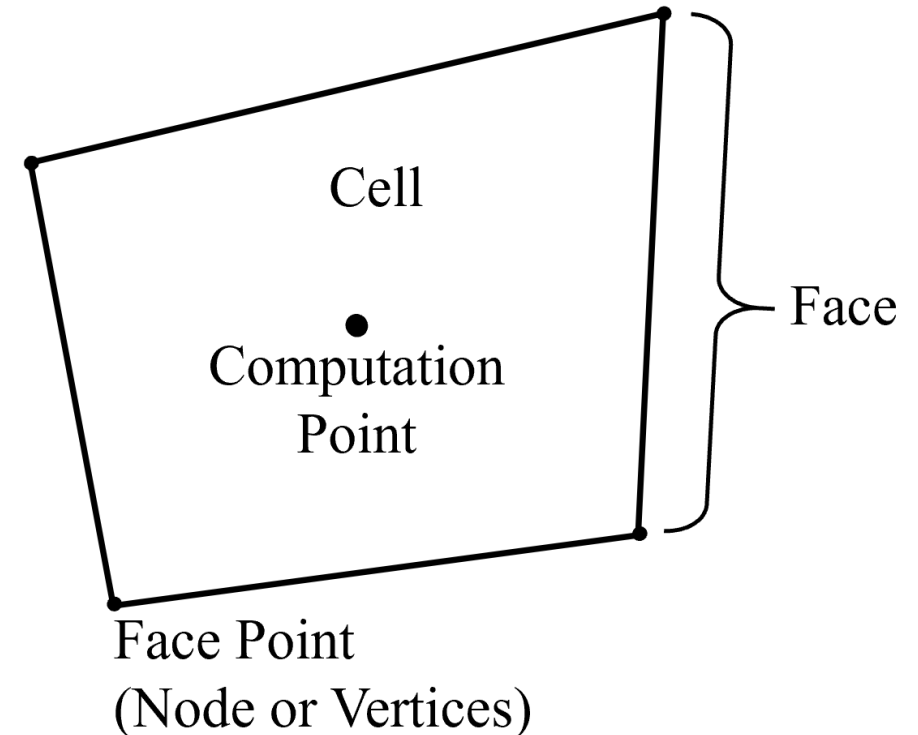


- Use SWE for:
 - Flows with dynamic changes in acceleration
 - Studies with important wave effects, tidal flows
 - Detail solution of flows around obstacles, bridges or bends
 - Simulations influenced by Coriolis, mixing, or wind
 - To obtain high-resolution and detailed flows
- Use DWE for:
 - Flow is mainly driven by gravity and friction
 - Fluid acceleration is monotonic and smooth, no waves
 - To compute approximate global estimates such as flood extent
 - To assess approximate effects of dam breaks
 - To assess interior areas due to levee breaches
 - For quick estimations or preliminary runs



Computational Mesh

- Mesh/grid can be unstructured
- Polygonal cells of up to 8 sides
- Cells must be concave
- Multiple 2D mesh can be run together or independently
- Grid Notation
 - Cells, Faces, Face Points (i.e. nodes or vertices), Computational Points, etc.
- State Variables
 - Cell Water levels
 - Face-normal Velocities





Numerical Methods

- Both DWE and SWE solvers are **Semi-implicit**
- Terms treated as:
 - Explicit: acceleration and diffusion terms
 - Semi-implicit: friction, flow divergence terms, and water level gradient
 - Fully-Implicit: pressure gradient term (for $\theta = 1$)
- By treating the “fast” pressure gradient term implicitly, the time step limitation based on the wave celerity can be removed
- Both DWE and SWE use **Finite-Difference** and **Finite-Volume** Methods
- Time integration: **Finite-Difference**
- Continuity Equation: **Finite-Volume**
- Momentum Equation: **Finite-Difference** (no control volume)



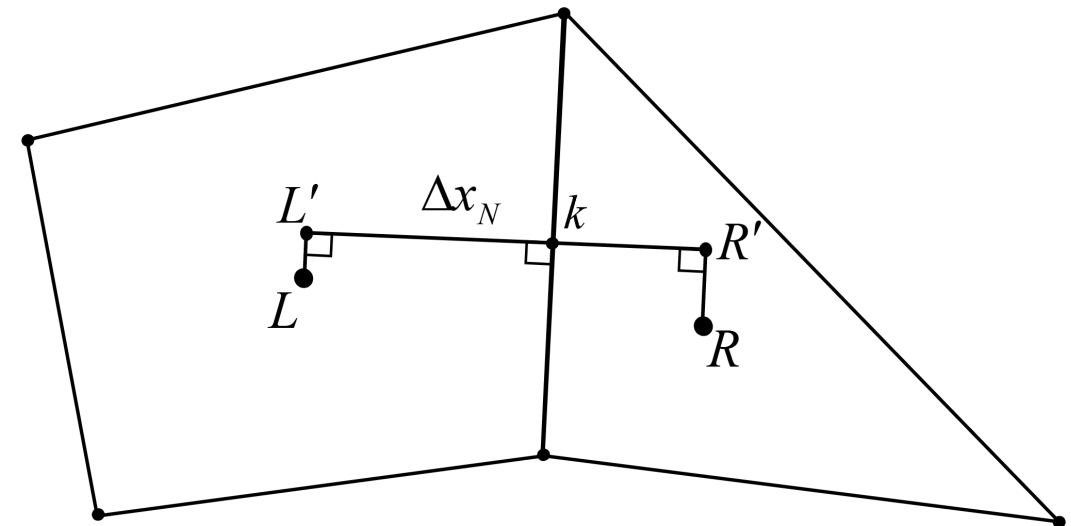
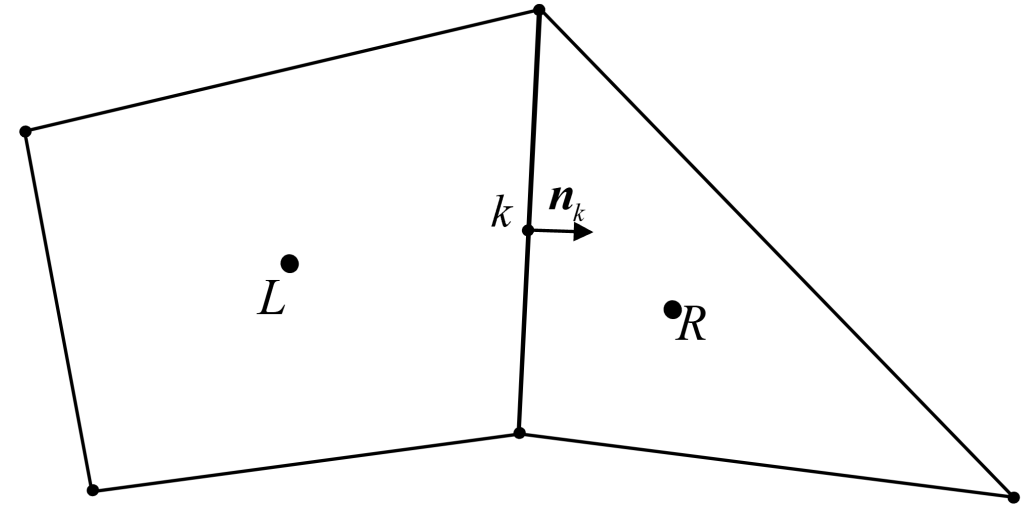
Face Water Surface Gradient



- Face-Normal Gradient

$$\nabla z_s \cdot \mathbf{n}_k = \frac{\partial z_s}{\partial N} \approx \frac{z_{s,R} - z_{s,L}}{\Delta x_N}$$

- Uses **Cell Centroids** and **NOT the Computation Points**
- Future versions may include non-orthogonal
- Compact two-point stencil is computationally efficient and robust
- Important to have a good quality mesh to reduce errors





Momentum Conservation

- Momentum conservation is directionally invariant
- Only “face-normal” component is needed at faces so

$$\frac{\partial u_N}{\partial t} + (\mathbf{V} \cdot \nabla) u_N - f_c u_T = -g \frac{\partial z_s}{\partial N} + \frac{1}{h} \nabla \cdot (\mathbf{v}_t h \nabla u_N) - \frac{\tau_{b,N}}{\rho R} + \frac{\tau_{s,N}}{\rho h}$$

where u_N is the velocity in the N direction

Face-Tangential Velocity

- Tangential velocities are computed on left and right of face with a Least-squares Formulation

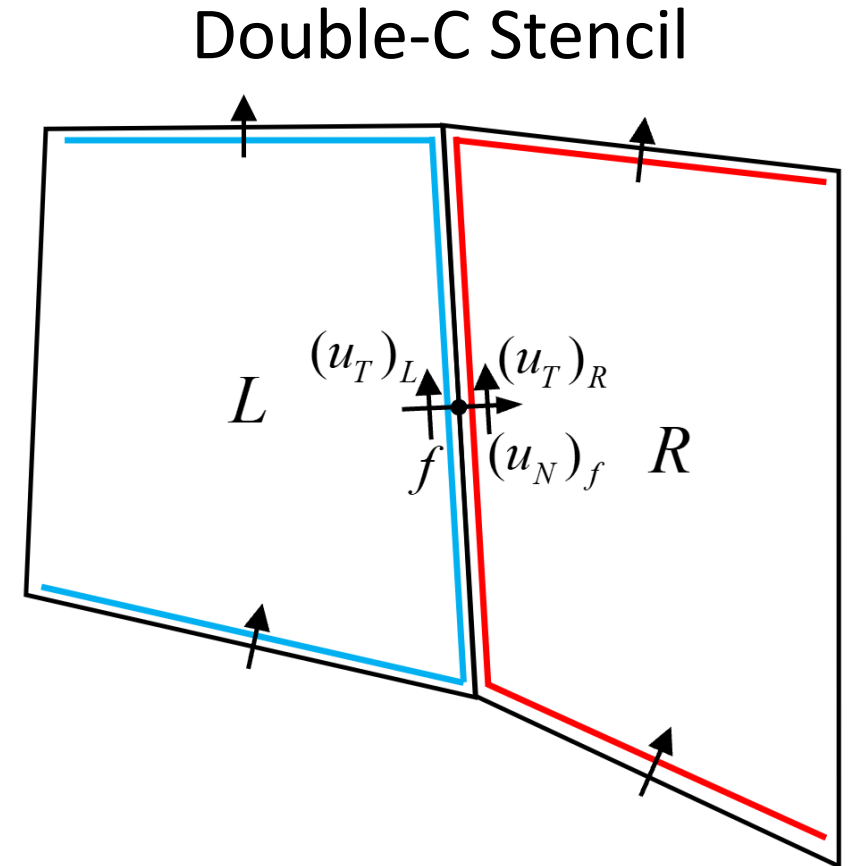
$$S_R = \sum_{k \in R} (\mathbf{V}_R \cdot \mathbf{n}_k - (u_N)_k)^2 \quad S_L = \sum_{k \in L} (\mathbf{V}_L \cdot \mathbf{n}_k - (u_N)_k)^2$$

- Of the left and right reconstructed velocities, only the tangential component is used, because the normal component is known

$$(u_T)_R = \mathbf{V}_R \cdot \mathbf{t}_f \quad (u_T)_L = \mathbf{V}_L \cdot \mathbf{t}_f$$

- Average face-tangential velocity computed as

$$(u_T)_f = \frac{(u_T)_R + (u_T)_L}{2}$$



Discretization

- Cell Velocity Gradient (x-direction)

- Gauss' Divergence Theorem

$$\nabla u_i = \frac{1}{A_i} \int_A \nabla u dA = \frac{1}{A_i} \oint_L u \mathbf{n} dL = \frac{1}{A_i} \sum_{k \in i} u_k \mathbf{n}_{ik} L_k$$

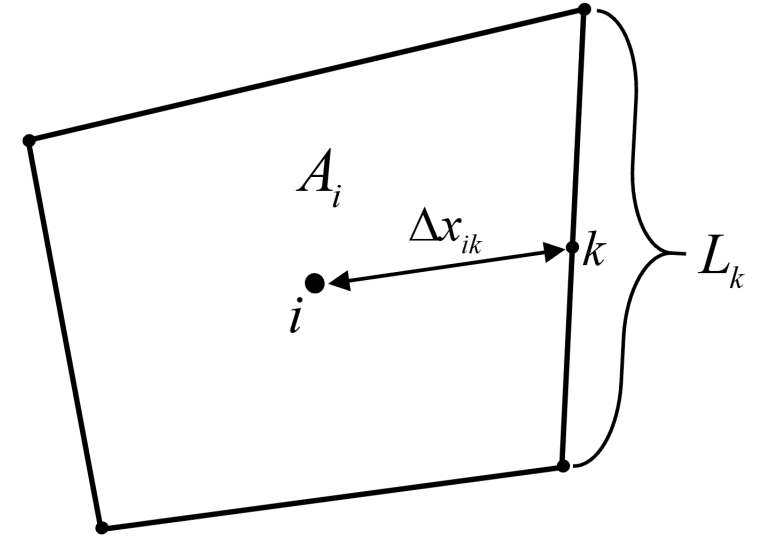
- Needed for turbulence modeling

- Cell Velocity

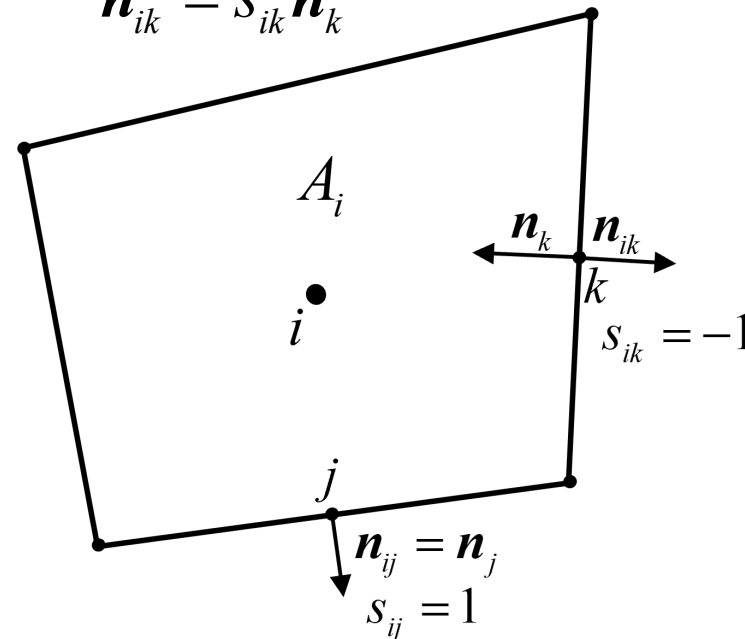
- Perot's Method

$$\mathbf{V}_i = \frac{1}{A_i} \sum_{k \in i} \Delta x_{ik} L_k \mathbf{n}_k (u_N)_k$$

- Needed for the conservative form of the mixing term and for Eulerian advection



$$\mathbf{n}_{ik} = S_{ik} \mathbf{n}_k$$





Discretization: Laplacian

- Node Laplacian

$$(\nabla^2 V)_j = [\nabla \cdot (\nabla V)]_j \approx \sum_i d_i (\nabla V)_i$$

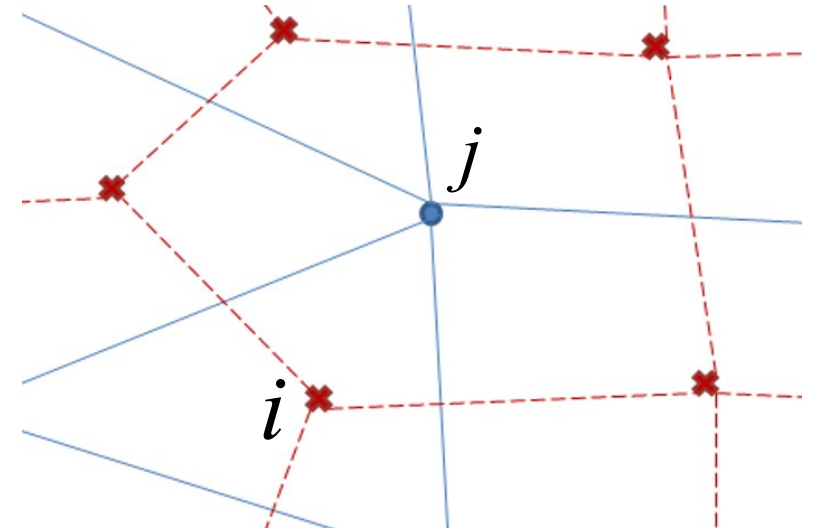
i : Cells

$$(\nabla V)_i = \sum_k c_k V_k$$

j : Nodes

k : Faces

- Used only by non-conservative turbulence



Backtracking

1. Interpolate node velocities from faces
2. Set starting location and remaining time as f and $T_R = \Delta t$
3. From starting location and velocity, find location B
4. Compute time to location B : $T_B = (\mathbf{x}_A - \mathbf{x}_B) \mathbf{V}_A^{-1}$
5. Interpolate velocity at location B : $\mathbf{V}_B = w_{n1} \mathbf{V}_{n1} + w_{n2} \mathbf{V}_{n2}$

if $T_B > T_R$

6. Set $A = B$, $T_R = T_R - T_B$, and go to step 3

else

7. Find location X as $\mathbf{x}_X = \mathbf{x}_f - T_R \mathbf{V}_A$

8. Interpolate velocity vector at X

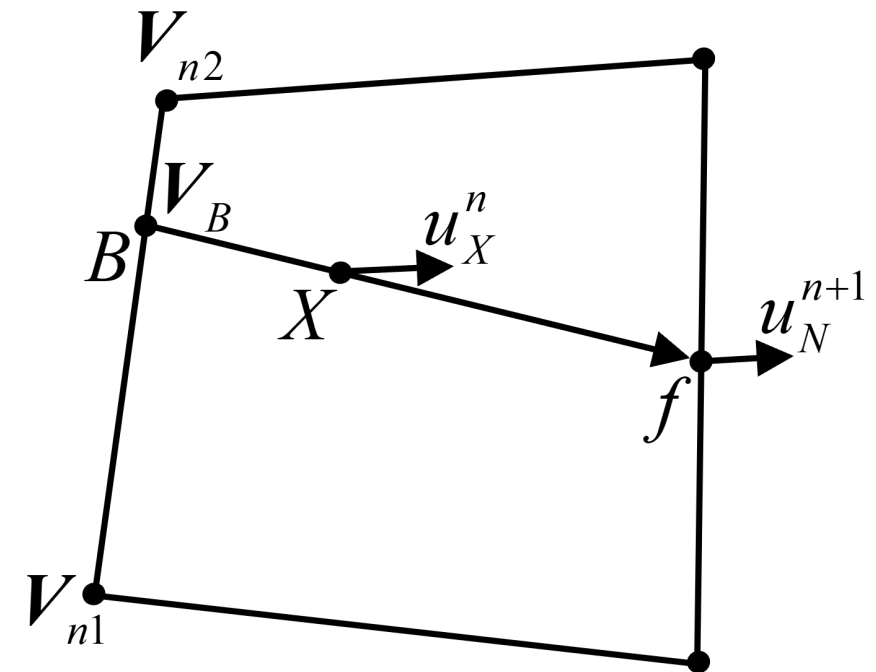
$$\mathbf{V}_X = T_B^{-1} \left[T_R \mathbf{V}_B + (T_B - T_R) \mathbf{V}_A \right]$$

9. Compute advective velocity

$$\mathbf{u}_X = \mathbf{n}_f \cdot \mathbf{V}_X$$

$$\frac{\partial u_N}{\partial t} + (\mathbf{V} \cdot \nabla) u_N = \frac{Du_N}{Dt} \approx \frac{u_N^{n+1} - u_X^n}{\Delta t}$$

$$\frac{d\mathbf{x}_P}{dt} = \mathbf{V}(\mathbf{x}, t)$$





Fractional Step Method (ELM only)

- Coriolis Term approximated as

$$f_c \mathbf{k} \times \mathbf{V} \approx \begin{pmatrix} f \left[(1-\theta) f v_X^n + \theta v^{n+1} \right] \\ -f \left[(1-\theta) f u_X^n + \theta u^{n+1} \right] \end{pmatrix}$$

where

f : Coriolis Parameter

θ : Implicit weighting factor

\mathbf{k} : Unit vector in the vertical direction

$\mathbf{V} = (u, v)^T$: Velocity at face

$\mathbf{V}_X = (u_X, v_X)^T$: Velocity at face at location X

- First (Coriolis) Step

$$\begin{pmatrix} 1 & \theta \Delta t f \\ \theta \Delta t f & 1 \end{pmatrix} \begin{pmatrix} u^* \\ v^* \end{pmatrix} = \begin{pmatrix} u_X^n + (1-\theta) \Delta t f v_X^n \\ v_X^n + (1-\theta) \Delta t f u_X^n \end{pmatrix} \quad \mathbf{V}^* = \begin{pmatrix} u^* \\ v^* \end{pmatrix}$$

- Second Step includes all other terms



Eulerian-Lagrangian Momentum Equation

- Semi-discrete form (2nd Fractional Step)

$$\frac{u_N^{n+1} - u_N^*}{\Delta t} = -g \frac{\partial z_s^{n+\theta}}{\partial N} + \left[\frac{1}{h} \nabla \cdot (v_t h^n \nabla u_N) \right]_X^n - c_f u_N^{n+1} + \frac{\tau_{s,N}}{\rho h_f^n}$$

where

$$z_a^{n+\theta} = (1 - \theta) z_s^n + \theta z_s^{n+1}$$

$$u_N^* = \mathbf{V}^* \cdot \mathbf{n}_f$$

- Velocity \mathbf{V}^* includes Coriolis
- Mixing term is interpolated at backtracking location X and based on previous time step velocity field
- Friction term is semi-implicit



Eulerian Momentum Equation

- Semi-discrete form

$$\frac{u_N^{n+1} - u_N^n}{\Delta t} + (\mathbf{V}^n \cdot \nabla) u_N^n - f u_T^n = -g \frac{\partial z_s^{n+\theta}}{\partial N} + \left[\frac{1}{h} \nabla \cdot (\mathbf{v}_t h \nabla u_N) \right]_f^n - c_f u_N^{n+1} + \frac{\tau_{s,N}}{\rho h_f^n}$$

where

$$z_s^{n+\theta} = (1 - \theta) z_s^n + \theta z_s^{n+1} \quad \bar{h}_f = \alpha_f^L h_L + \alpha_f^R h_R$$

- Coriolis term computed at face f and is explicit
- No fractional step method like ELM solver
- Mixing term is computed at face f and is explicit
- Friction and pressure gradient terms are semi-implicit



Discretization: Eulerian Advection

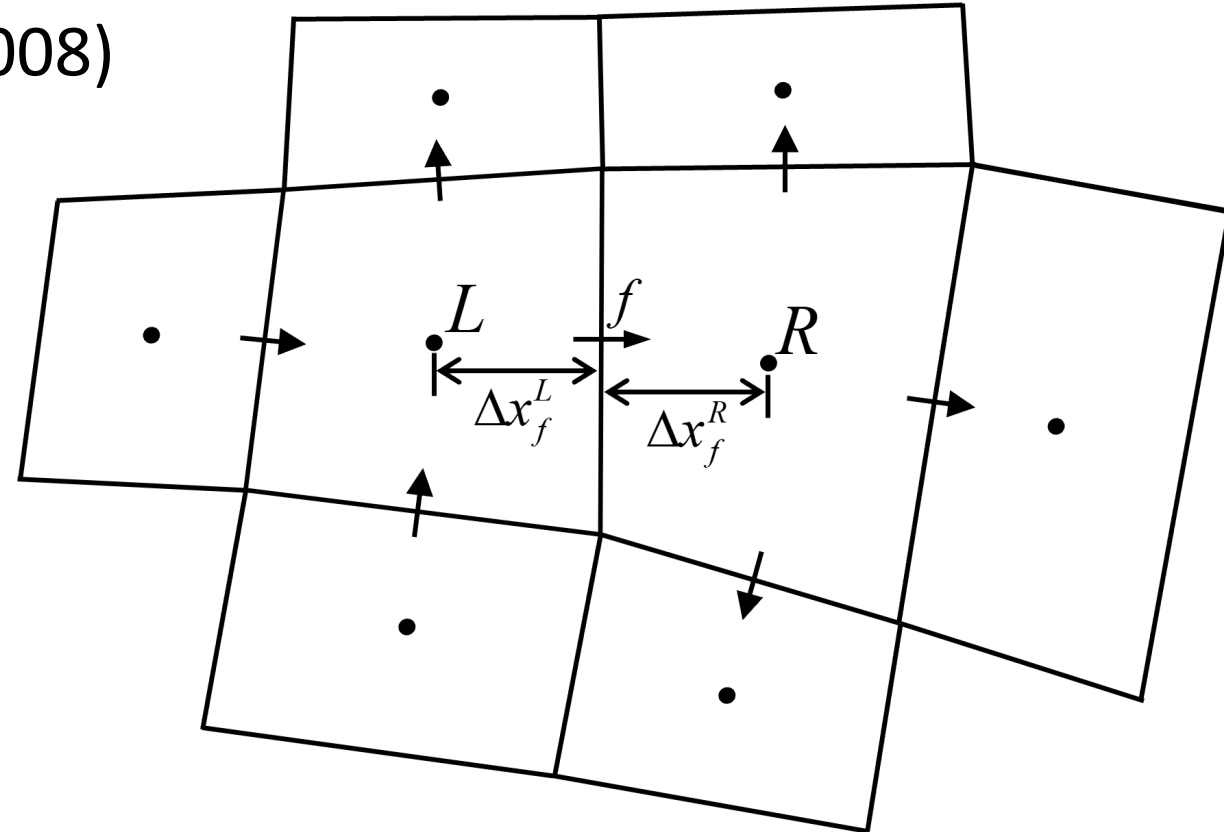
- Approach from Kramer and Stelling (2008)

$$(\mathbf{V} \cdot \nabla) u_N \approx \frac{\alpha_f^L}{\bar{h}_f A_L} \sum_{k \in L} s_{Lk} Q_k \left[\mathbf{V}_k^u \cdot \mathbf{n}_f - (u_N)_f \right]$$

$$+ \frac{\alpha_f^R}{\bar{h}_f A_R} \sum_{k \in R} s_{Rk} Q_k \left[\mathbf{V}_k^u \cdot \mathbf{n}_f - (u_N)_f \right]$$

$$\alpha_f^L = \frac{\Delta x_f^L}{\Delta x_f^L + \Delta x_f^R} \quad \bar{h}_f = \alpha_f^L h_L + \alpha_f^R h_R$$

$$\alpha_f^R = 1 - \alpha_f^L$$



- Courant-Freidrichs-Lewy (CFL) Condition

$$C = \frac{U \Delta t}{\Delta x} \leq 1$$



Discretization: Mixing Term

- Non-Conservative Form

$$\mathbf{v}_t \nabla^2 u_N \Big|_f \approx v_{t,f}^n \left(\nabla^2 V \right)_X^n \cdot \mathbf{n}_f$$

- Conservative Form

$$\frac{1}{h} \nabla \cdot (\mathbf{v}_t h \nabla u_N) \Big|_f \approx \frac{\alpha_f^L}{\bar{h}_f A_L} \sum_{k \in L} A_k v_{t,k} \frac{\mathbf{n}_f \cdot (V_j - V_L)}{\Delta x_{L,j}} + \frac{\alpha_f^R}{\bar{h}_f A_R} \sum_{k \in R} A_k v_{t,k} \frac{\mathbf{n}_f \cdot (V_j - V_R)}{\Delta x_{R,j}}$$

- Discretization same for both ELM and EM solvers
- Approximate Stability Criteria for EM solver

$$\frac{v_t \Delta t}{\Delta x^2} \leq \frac{1}{2}$$

- ELM interpolates term to location X

$$\left[\frac{1}{h} \nabla \cdot (\mathbf{v}_t h \nabla u_N) \right]_X^n$$



Why is the Temporal Term Important?

- Assuming deep water, no friction or other forces

$$\frac{\partial z_s}{\partial t} + h \frac{\partial u}{\partial x} = 0 \qquad \frac{\partial u}{\partial t} + g \frac{\partial z_s}{\partial x} = 0$$

- Eliminating the velocity leads to the **classical wave equation**

$$\frac{\partial^2 z_s}{\partial t^2} = c^2 \frac{\partial^2 z_s}{\partial x^2} \qquad c = \sqrt{gh} \qquad c : \text{Celerity}$$

- Temporal term is necessary for **wave propagation**



Eulerian-Lagrangian vs. Eulerian SWE Solvers

- **SWE-ELM**

- Only solver available in V5.0.7 and earlier
- Default in V6.0
- Not limited by Courant condition
- Excellent stability
- Can have momentum conservation problems around shocks or where the flow changes rapidly

- **SWE-EM**

- New to V6.0 as an option
- Limited to Courant less than 1.0
- Good Stability
- Improved momentum conservation for all flow conditions

Strength/Feature/Capability	SWE-ELM	SWE-EM
Larger Time Step	X	
Best Stability	X	
Courant Stability Criteria		X
Diffusion Stability Criteria		X
Computational Speed	X	
Wet/dry > 1 cell per time step	X	
Best Momentum Conservation		X
Non-Conservative Mixing		X
Conservative Mixing	X	X
Wind	X	X



SWE-ELM and SWE-EM vs SWE-LIA

- SWE-ELM and SWE-EM
 - Include momentum advection term
 - Differ in their approach to computing momentum advection term
 - Slower
 - Less stable
 - Include temporal term
- SWE-LIA
 - Ignores the advection term
 - Faster (momentum term can be 20-30% of run time)
 - More stable
 - Also includes the temporal term

Strength/Feature/Capability	SWE-ELM SWE-EM	SWE-LIA
Temporal Term	X	X
Advection Term	X	
Larger Time Step		X
Best Stability		X
Wave propagation	X	X
Diffusion Term	X	
Wind	X	X
Atmospheric pressure	X	X



Solution Procedure



- System of equations

$$\mathbf{\Omega} + \mathbf{\Psi Z} = \mathbf{b}$$

- Algorithm

1. Compute Right-Hand-Side \mathbf{b}
 - Contains explicit terms:
advection, diffusion, wind, etc.
2. Outer Loop (Assembly and Updates)
 - Update linearized terms and variables
including coefficient matrix $\mathbf{\Psi}$
3. Inner Loop (Newton Iterations)

$$\mathbf{Z}^{m+1} = \mathbf{Z}^m - [\mathbf{\Psi} + \mathbf{A}^m]^{-1} (\mathbf{\Omega}^m + \mathbf{\Psi Z}^m - \mathbf{b})$$

\mathbf{Z} : Water level

$\mathbf{\Omega}$: Water volume

$\mathbf{\psi}$: Coefficient matrix

\mathbf{b} : Right-hand-side

m : Iteration index

\mathbf{A} : Diagonal matrix of
cell wet surface areas



Boundary Conditions

- **Stage Hydrograph.** Upstream or downstream
- **Flow Hydrograph.** Upstream or downstream. Local conveyance and velocities computed automatically.
- **Normal Depth BC.** At downstream boundaries.
- **Rating Curve BC.**
- **Wind.** Only for shallow-water equations.
- **Precipitation, evapotranspiration, and infiltration.** Included as sources and sinks in the continuity equation.
- 1D reaches and 2D areas can be connected
- Multiple 2D areas can be connected to each other
- 2D areas can be connected to 1D lateral structures such as levees to simulate levee breaches



Computational Implementation



- Multiple 2D areas can be computed independently and simultaneously
- All solvers are can be run on multiple cores
- 2D solvers and parameters can be selected independently for each 2D area
- A partial grid solution keeps track of active portion of mesh and only computes the solution for active portion significantly reducing computational times.

Thank You!

HEC-RAS Website:

<https://www.hec.usace.army.mil/software/hec-ras/>

Online Documentation:

<https://www.hec.usace.army.mil/confluence/rasdocs>