HEC-RAS 2D Sediment Advection-Diffusion Parameters

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Concentration Definition

- **Depth-averaged**

$$\hat{C}_{tk} = \frac{1}{h}\int_0^h c_{tk}\,dz \qquad q_{tk} = \beta_t U h \hat{C}_{tk} \qquad \beta_t = \frac{1}{Uh\hat{C}_{tk}}\int_0^h u c_{tk}\,dz$$

  - Coefficient for transport (advection term)
  - **Used in HEC-RAS 1D**
- **Velocity weighted** (Einstein definition)

$$C_{tk} = \frac{1}{Uh}\int_0^h u c_{tk}\,dz \qquad q_{tk} = UhC_{tk}$$

  - Simpler formula for transport (advection term)
  - **Used in HEC-RAS 2D**
  - Coefficient in temporal term

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Total-load Transport Equation

- Unsteady Advection-Diffusion Coefficient

$$\underbrace{\frac{\partial}{\partial t}\left(\frac{hC_{tk}}{\beta_{tk}}\right)}_{\text{Temporal or Storage}} + \underbrace{\nabla\cdot\left(h\mathbf{U}C_{tk}\right)}_{\text{Advection}} = \underbrace{\nabla\cdot\left(\boldsymbol{\varepsilon}_{tk} h\nabla C_{tk}\right)}_{\text{Diffusion}} + \underbrace{E_{tk}}_{\text{Erosion}} - \underbrace{D_{tk}}_{\text{Deposition}}$$

k : Grain class
h : Water depth
C_{tk} : Total-load concentration
β_{tk} : Total-load correction factor
$\mathbf{U}$: Current velocity
$\boldsymbol{\varepsilon}_{tk}$: Total-load diffusion coefficient
E_{tk} : Total-load erosion rate
D_{tk} : Total-load deposition rate

- Simulating total-load instead of separate bed- and suspended-loads reduces computational costs because it requires half as many transport equations

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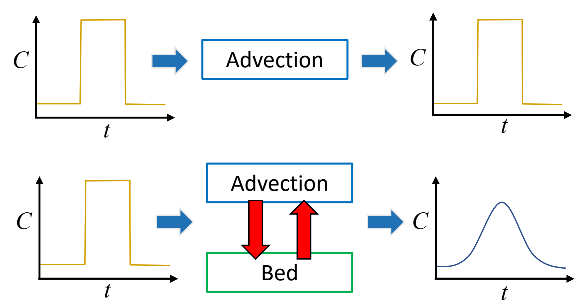
Background

- Most natural conditions are **advection-dominated**
- Diffusion is utilized here to describe the horizontal mixing of sediment **in the water column** due to turbulent mixing and dispersion
- **Dispersion** of bed material also occurs due to the bed mixing
  - Bed provides a storage mechanism

Sediment Concentration Time-Series



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Diffusion Coefficients

- Total-load diffusion coefficient

$$\boldsymbol{\varepsilon}_{tk} = r_{sk}\boldsymbol{\varepsilon}_{sk} + (1 - r_{sk})\varepsilon_{bk}$$

- Suspended-load diffusion coefficient

$$\boldsymbol{\varepsilon}_{sk} = \frac{\mathbf{v}_t}{\sigma_{sk}} \qquad \boldsymbol{\varepsilon}_{sk} = \begin{bmatrix} \boldsymbol{\varepsilon}_{s,xx} & \boldsymbol{\varepsilon}_{s,xy} \\ \boldsymbol{\varepsilon}_{s,yx} & \boldsymbol{\varepsilon}_{s,yy} \end{bmatrix}$$

  - Eddy viscosity from hydraulic model
    or
  - Computed as

$$\nu_t = c_M u_* h$$

- Bed-load diffusion coefficient

$$\varepsilon_{bk} = c_B u'_* d_k$$

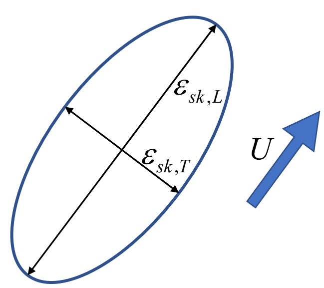
ε_{lk} : Diffusion coefficient [L^2/T]

r_{sk} : Fraction of suspended load [-]

σ_{sk} : Schmidt Number [-]

u_* : Shear velocity [L/T]

c_M, c_B : Calibration oefficients [-]



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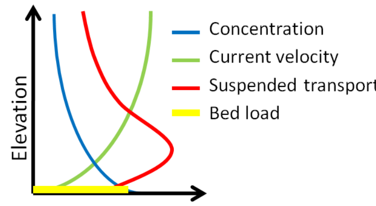
Load-Correction Factors

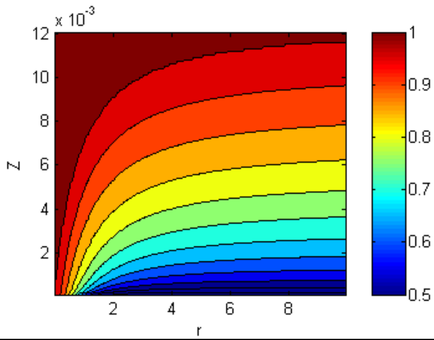
- Total-load Correction Factor

$$\beta_{tk} = \frac{1}{r_{sk}/\beta_{sk} + (1-r_{sk})/\beta_{sk}}$$

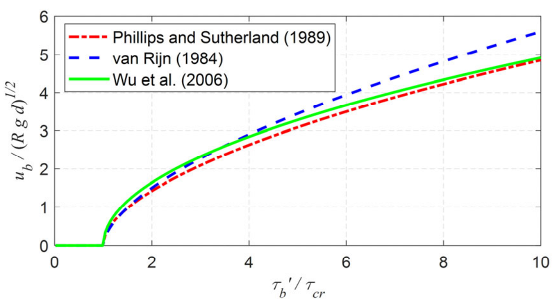
- Suspended-load

$$\beta_{sk} = \frac{\int_0^h u c_k dz}{U C_k}$$





- Bed-load $\beta_{bk} = \frac{u_{bk}}{U}$



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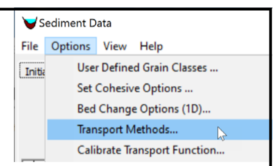
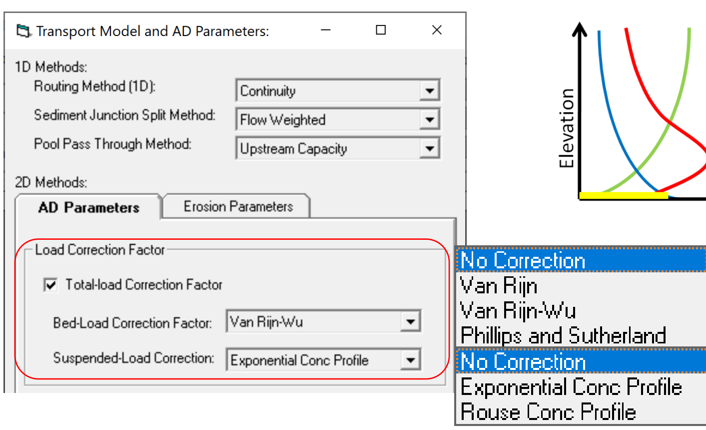
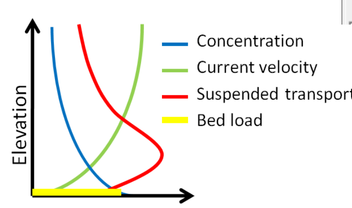
Total-Load Correction Factor







Transport equation

$$\frac{\partial}{\partial t}\left(\frac{hC_{tk}}{\beta_{tk}}\right) + \nabla \cdot (h\boldsymbol{U}C_{tk}) = \nabla \cdot \left(\varepsilon_{tk} h \nabla C_{tk}\right) + E_{tk} - D_{tk}$$

$$\beta_{tk} = \frac{1}{r_{sk}/\beta_{sk} + (1-r_{sk})/\beta_{bk}}$$

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Fraction of Suspended Sediments

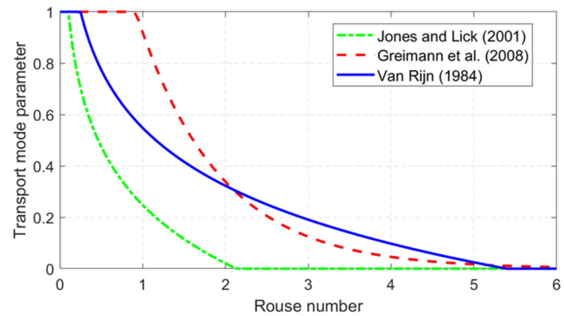
- Defined as ratio of suspended to total-load

$$r_{sk} = \frac{q_{sk}}{q_{tk}}$$

- Approximated as ratio of potential loads

$$r_{sk} \approx f_{sk} = \frac{q_{sk}^*}{q_{tk}^*}$$

  - Methods
    - Capacity
    - Empirical methods



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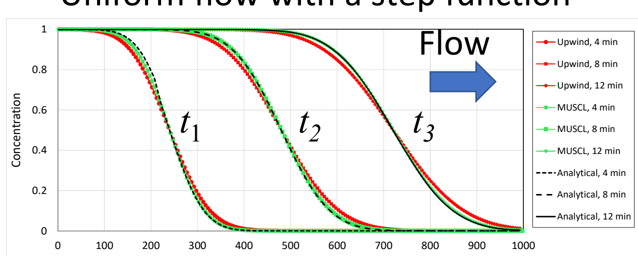
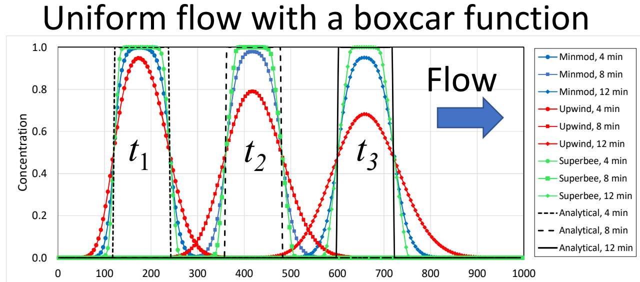
Verification: Advection and Diffusion

- Analytical Problems
  - Grid and time step convergence
  - Analysis of relative performance of difference schemes

Uniform flow with a step function



Uniform flow with a boxcar function



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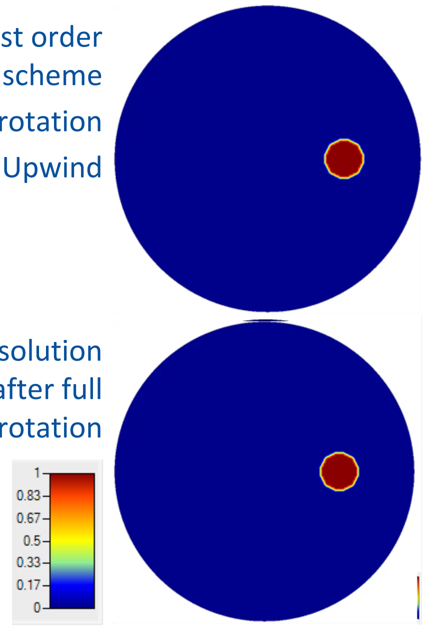
Verification: Advection

- Advection of a scalar in a circular basin
- Purpose to evaluate numerical diffusion, monotonicity, and conservation
- Scalar should rotate without changing
  shape or exceeding initial bounds

First order scheme
after full rotation
Upwind

High-resolution scheme after full rotation



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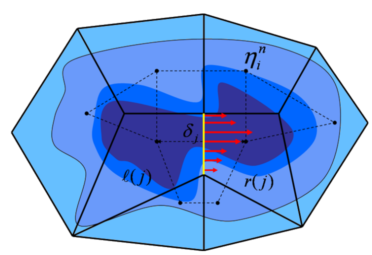
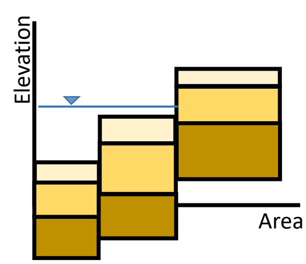
Overview of Numerical Methods

- Transport Equation
  - Finite-Volume Methods
  - Implicit Generalized Euler Temporal Scheme (1^{st} – 2^{nd} Order)
  - Advection Schemes (1^{st} and 2^{nd} Order)
    - Linear 1^{st} Order Difference Schemes
    - Nonlinear 2^{nd} Order Flux Limiter Schemes
  - Anisotropic Diffusion
    - Linear Two-Point Flux Approximation
  - Direct and Iterative Sparse Matrix Solvers
- Semi-Coupled Flow-Sediment Scheme
- Bed Sorting and Layering
  - Finite-Differences
- **Subgrid Hydrodynamics, Erosion, Deposition, Bed Change, Sorting, and Layering**





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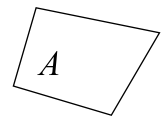
Numerical Methods

- Advection-Diffusion Equation

$$\underbrace{\frac{\partial}{\partial t}\left(\frac{hC_{tk}}{\beta_t}\right)}_{\text{Temporal or Storage}} + \underbrace{\nabla\cdot\left(h\mathbf{U}C_{tk}\right)}_{\text{Advection}} = \underbrace{\nabla\cdot\left(\boldsymbol{\varepsilon}_{tk}h\nabla C_{tk}\right)}_{\text{Diffusion}} + \underbrace{E_{tk}}_{\text{Erosion}} - \underbrace{D_{tk}}_{\text{Deposition}}$$



- Integral Form

$$\int_A \frac{\partial}{\partial t}\left(\frac{hC_{tk}}{\beta_t}\right)dA + \int_A \nabla\cdot\left(h\mathbf{U}C_{tk}\right)dA = \int_A \nabla\cdot\left(\boldsymbol{\varepsilon}_{tk}h\nabla C_{tk}\right)dA + \int_A \left(E_{tk} - D_{tk}\right)dA$$

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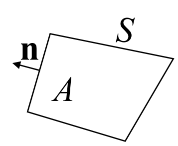
Discretization: Diffusion

- Finite-Volume Discretization

$$\int_A \nabla\cdot\left(\boldsymbol{\varepsilon}_{tk}h\nabla C_{tk}\right)\mathrm{d}A = \int_S \left[\left(\boldsymbol{\varepsilon}_{tk}h\nabla C_{tk}\right)\cdot\mathbf{n}\right]\mathrm{d}S \approx \sum_{f\in S_c} A_f\left(\boldsymbol{\varepsilon}_{tk}\nabla C_{tk}\right)_f\cdot\mathbf{n}_f$$



- Linear Two-Point Flux Approximation
  - To simplify notation let $\phi = C_{tk}$
  - then

$$\left(\boldsymbol{\varepsilon}\nabla\phi\right)_f\cdot\mathbf{n}_f \approx \left[\left(\boldsymbol{\varepsilon}_f\mathbf{n}_f\right)\cdot\mathbf{n}_{PN}\right]\left(\nabla\phi_f\cdot\mathbf{n}_{PN}\right) \approx \varepsilon_f\nabla_\perp C_{tk,f} \approx \varepsilon_f\frac{\phi_N-\phi_P}{\Delta x_{PN}}$$

$$\Delta x_{PN} = \left|\boldsymbol{x}_P - \boldsymbol{x}_N\right|$$

$$\varepsilon_f = \left(\boldsymbol{\varepsilon}\mathbf{n}_f\right)\cdot\mathbf{n}_{PN}$$

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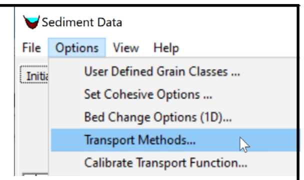
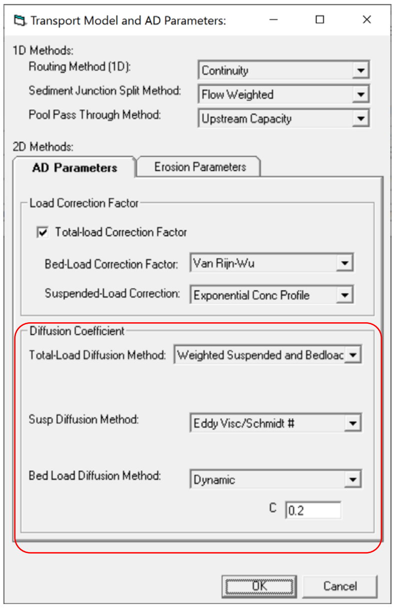
Diffusion Coefficient

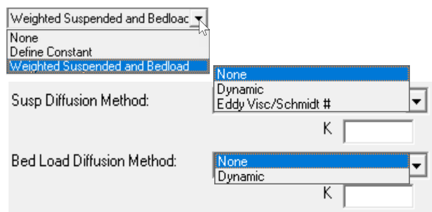




Transport equation

$$\frac{\partial}{\partial t}\left(\frac{hC_{tk}}{\beta_{tk}}\right)+\nabla\cdot(h\boldsymbol{U}C_{tk})=\nabla\cdot\left(\varepsilon_{tk}h\nabla C_{tk}\right)+E_{tk}-D_{tk}$$

- Accounts for:
  - Turbulent mixing
  - Dispersion
- **Dynamic** requires a coefficient



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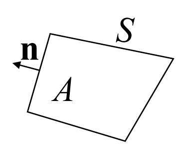
Discretization: Advection

- Finite-Volume Discretization

$$\int_A \nabla\cdot\left(h\mathbf{U}C_{tk}\right)dA=\int_S\left[\left(h\mathbf{U}C_{tk}\right)\cdot\mathbf{n}\right]dS\approx\sum_{f\in S_c}s_{c,f}Q_f C_{tk,f}$$



- Advected concentrations at faces are interpolated from cell values with advection scheme
  - Upwind
  - Exponential
  - Harmonic
  - Minmod

$s_{c,f}=\mathbf{n}_{c,f}\cdot\mathbf{n}_f$

$s_{c,f}$: Outward orientation [-]

Q_f : Face Flow [L^2/T]

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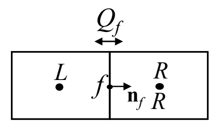
Advection Schemes

- Upwind Scheme

$$\phi_f = \phi_C = \frac{1}{Q_f}\left(Q_f^+\phi_L + Q_f^-\phi_R\right) \qquad Q_f^\pm = \frac{1}{2}\left(Q_f \pm |Q_f|\right)$$



  - First-order
  - Most Stable

$s_{c,f} = \mathbf{n}_{c,f} \cdot \mathbf{n}_f$

$s_{c,f}$: Outward orientation [-]

- Advection Term

$$\sum_{f \in K(c)} s_{c,f} Q_f \phi_f = \sum_{f \in K(c)} s_{c,f}\left(Q_f^+\phi_L + Q_f^-\phi_R\right)$$

$Q_f = A_f U_f$: Face flow [L^3/T]

A_f : Face area [L^2]

U_f : Face velocity [L/T]

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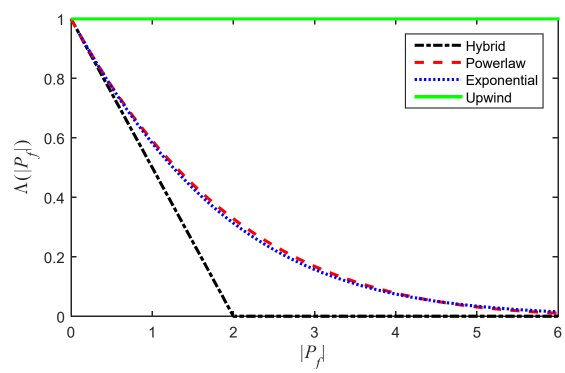
Advection Schemes

- Exponential Scheme
  - Based on 1D analytical solution of AD eq.
  - Linear and 1st order
  - Reduced diffusion by applying a reduction to the mixing term
  - Reduces to upwind scheme if there is no diffusion term



$$Q_f\phi_f - A_f\varepsilon_{t,f}\nabla_\perp\phi_f = \left[D_f\Lambda\left(|P_f|\right) + Q_f^+\right]\phi_L - \left[D_f\Lambda\left(|P_f|\right) - Q_f^-\right]\phi_R$$

$$\Lambda\left(|P_f|\right) = \frac{|P_f|}{\exp\left(|P_f|\right) - 1} \qquad D_f = \frac{A_f\varepsilon_{t,f}}{\Delta x_{PN}} \qquad P_f = \frac{U_f \Delta x_{PN}}{\varepsilon_f} \qquad Q_f^\pm = \frac{1}{2}\left(Q_f \pm |Q_f|\right)$$

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Advection Schemes

- High-Resolution Schemes

$$\phi_f = \phi_C + \frac{\psi(r)}{2}(\phi_D - \phi_C)$$

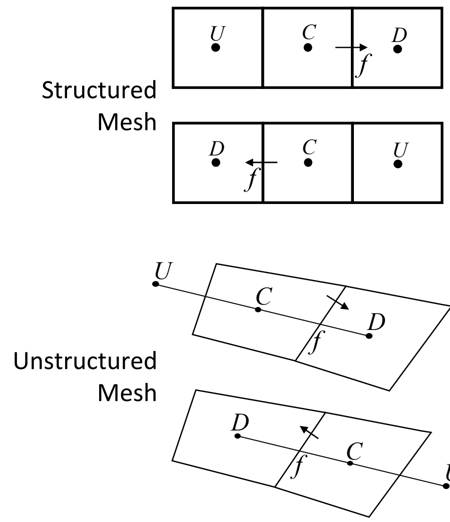
  - Sweby r-factor

$$r = \frac{\Delta x_{CD}}{\phi_D - \phi_C}\frac{\phi_C - \phi_U}{\Delta x_{UC}}$$

  - Second upstream value estimated as

$$\phi_U = \phi_D - 2\nabla\phi_C \cdot \mathbf{r}_{CD}$$

$$\mathbf{r}_{CD} = \mathbf{x}_D - \mathbf{x}_C$$



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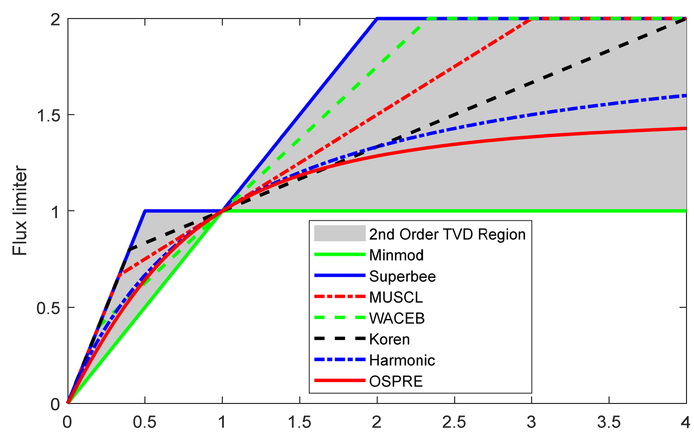
Advection Schemes

- Flux Limiters
  - Minmod (Roe and Bains 1982)

$$\psi(r) = \max\left[0, \min(r,1)\right]$$

  - Harmonic (van Leer 1974)

$$\psi(r) = \frac{r + |r|}{1 + |r|}$$



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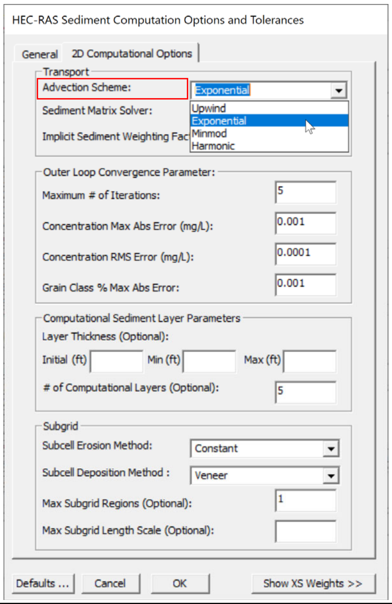
Advection Scheme



- Upwind
  - Most stable (and diffusive)
  - First order and Linear (no iterations)
- Exponential (Patankar 1980)
  - Based on 1D steady solution of Advection-Diffusion Equation
  - First Order and linear (no iterations)
- Minmod (Roe 1985)
  - TVD Flux Limiter
  - Second Order
  - Non-linear (requires iterations)
- Harmonic (van Leer 1977)
  - TVD Flux Limiter
  - Second Order
  - Non-linear (requires iterations)

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Discretization: Temporal Term

- Finite-Volume Discretization

$$\int_A \frac{\partial}{\partial t}\left(\frac{hC_{tk}}{\beta_{tk}}\right) \mathrm{d}A = \frac{\partial}{\partial t}\int_\Omega \frac{C_{tk}}{\beta_{tk}} \mathrm{d}\Omega = \frac{\partial}{\partial t}\left(\frac{\Omega_i C_{tk,i}}{\beta_{tk.i}}\right) = \frac{\Omega_i C_{tk,i}}{\Delta t \beta_{tk.i}} - \frac{\Omega_i C_{tk,i}}{\Delta t \beta_{tk.i}}$$

- Time stepping
  - Two-step method
  - Generalized Euler for advection-diffusion, and backward Euler for Erosion and Deposition

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Discretization

- Final Discrete Form

$$\frac{\Omega_i^{n+1}C_{tk,i}^{n+1}}{\beta_{tk,i}^{n+1}} = \frac{\Omega_i^{n}C_{tk,i}^{n}}{\beta_{tk,i}^{n}} - \Delta t C_{tk,i}^{n+\theta} \sum_{f \in \mathrm{K}(i)} \left[D_f \Lambda\left(\left|P_f\right|\right) + s_{i,f} Q_f^{+} \right] + \Delta t \sum_{\substack{f \in \mathrm{K}(i) \\ j \in \mathrm{N}(i)}} \left[D_f \Lambda\left(\left|P_f\right|\right) - s_{i,f} Q_f^{-} \right] C_{tk,j}^{n+\theta}$$

$$+ \Delta t \left(E_{tk,i}^{n+1} + \varpi_{tk,i} C_{tk,i}^{n+1} \right) A_i - \frac{\Delta t}{2} \sum_{\substack{f \in \mathrm{K}(i) \\ j \in \mathrm{N}(i)}} \left|Q_f\right| \psi(r) \left(C_{tk,j}^{n+1} - C_{tk,i}^{n+1} \right)$$

- Assembly

$$a_{i,i} C_{tk,i}^{n+1} + \sum_{j \in N(i)} a_{i,j} C_{tk,j}^{n+1} = b_i$$

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Recomendations

- Default advection scheme is **Exponential**, which reduces to **Upwind** if no diffusion is included
- Never use **Upwind** scheme and **diffusion** at the same time as this will produce too much diffusion
- If model convergence is good, switch to **High-Resolution** (i.e. **Harmonic** and **Minmod**) schemes or better accuracy and compare
- If **High-Resolution** scheme results not significantly different, switch back to **Exponential** scheme
- Use suspended diffusion coefficient based on turbulent eddy viscosity

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Thank You!

HEC-RAS Website:
https://www.hec.usace.army.mil/software/hec-ras/

Online Documentation:
https://www.hec.usace.army.mil/confluence/rasdocs

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